18.152 Midterm assignment solutions

Proof of Problem 1. $u(x) = \frac{1}{4}(1-|x|^2) \in C^\infty(\overline{\Omega})$ satisfies $\Delta u = -1$ in $\overline{\Omega}$ and $u = 0$ on $\partial\Omega$. Hence, the Green's representation formula implies the desired result. □

Proof of Problem 4. By Theorem 3, we have

$$u_i(x) - w_\epsilon(x) = \int_\Omega (\rho_\epsilon - 1)\Phi_{x_i}(x-y)f(y)dy \tag{1}$$

$$= \int_{B_{2\epsilon}(x)} (\rho_\epsilon - 1)\Phi_{x_i}(x-y)f(y)dy. \tag{2}$$

Since $|\Phi_{x_i}(x-y)| \leq C_1|x-y|^{-1}$, we have

$$|u_i - w_\epsilon| \leq C_2 \int_{B_{2\epsilon}(x)} |x-y|^{-1}dy = C_2 \int_0^{2\epsilon}\int_0^{2\pi} drd\theta = C_3\epsilon. \tag{3}$$

Next, we have

$$\frac{\partial}{\partial x_j} w_\epsilon(x) = -\int_\Omega \frac{\partial}{\partial x_j}\left[\rho_\epsilon(x,y)\frac{\partial}{\partial x_i}\Phi(x-y)\right](f(y)-f(x))dy \tag{4}$$

$$+ \int_\Omega \left(\frac{\partial^2}{\partial x_i \partial x_j}\varphi(x,y)\right) f(y)dy \tag{5}$$

$$- f(x)\int_\Omega \frac{\partial}{\partial x_j}\left[\rho_\epsilon(x,y)\frac{\partial}{\partial x_i}\Phi(x-y)\right] dy. \tag{6}$$

To reformulate the last integral, we observe

$$\frac{\partial}{\partial x_j}\big(\rho_\epsilon(x,y)\frac{\partial}{\partial x_j}\Phi(x-y)\big) = \frac{\partial}{\partial y_j}\big(\rho_\epsilon(x,y)\frac{\partial}{\partial y_j}\Phi(x-y)\big). \tag{7}$$

Since $\rho_\epsilon(x,y)\frac{\partial}{\partial y_j}\Phi(x-y) \in C^\infty(\overline{\Omega})$, Theorem 2 and $\rho_\epsilon(x,y) = 1$ on $\partial\Omega$ yield

$$\int_\Omega \frac{\partial}{\partial x_j}\left[\rho_\epsilon(x,y)\frac{\partial}{\partial x_i}\Phi(x-y)\right] dy = \int_{\partial\Omega}\left[\frac{\partial}{\partial y_i}\Phi(x-\sigma)\right]\nu_j(\sigma)d\sigma. \tag{8}$$

Hence,

$$v_{ij}(x) - \frac{\partial}{\partial x_j}w_\epsilon(x) = \int_{B_{2\epsilon}(x)} \frac{\partial}{\partial x_j}\left[(\rho_\epsilon - 1)\frac{\partial}{\partial x_i}\Phi(x-y)\right](f(y)-f(x))dy. \tag{9}$$

By using $|\nabla\rho_\epsilon| \leq C\epsilon^{-1}$, $|\nabla\Phi| \leq C|x-y|^{-1}$, $|\nabla^2\Phi| \leq C|x-y|^{-2}$, and $|f(x)-f(y)| \leq C|x-y|$, we have

$$|v_{ij} - \frac{\partial}{\partial x_j}w_\epsilon| \leq C_4 \int_{B_{2\epsilon}(x)} \epsilon^{-1} + |x-y|^{-1}dy \leq C_5\epsilon. \tag{10}$$

□

Proof of Problem 6. Everybody proved the existence. Hence, we show the uniqueness here.

Suppose that there exists two solutions $u, v \in D^2(\Omega) \cap C^0(\overline{\Omega})$. Then, $w = u - v \in D^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $\Delta w = 0$ in Ω and $w = 0$ on $\partial\Omega$.

We define $w_\epsilon(x) = \epsilon + \epsilon(1 - |x|^2)$ and claim $w \le w_\epsilon$ holds in Ω for all $\epsilon > 0$. If not, there exists a point $x_0 \in \Omega$ such that $w(x_0) - w_\epsilon(x_0) = \max_{\overline{\Omega}}(w - w_\epsilon) > 0$. Then, $\hat{w}_\epsilon = w - w_\epsilon$ attains its maximum at the interior point x_0, and thus $\Delta \hat{w}_\epsilon(x_0) \le 0$. However, $\Delta \hat{w}_\epsilon = \Delta w - \Delta w_\epsilon = 4\epsilon > 0$. Therefore, we have $w \le w_\epsilon$ for all $\epsilon > 0$. Passing $\epsilon \to 0$ yields $w \le 0$ in $\overline{\Omega}$. Similarly, we obtain $w \ge 0$ in $\overline{\Omega}$, and thus $w = 0$. □

First proof of Problem 7. We recall $a_0(r) = (2\pi)^{-1} \int_0^{2\pi} u(r,\theta) d\theta$ which satisfies $a_0'' + r^{-1} a_0' = 0$. Hence, $a_0 = c_1 + c_2 \log r$ for some constant c_1, c_2. By $a_0(r) > 0$, we have $c_2 = 0$ and thus $a_0(r) = c_1 > 0$.

Since c_1 is the average of u on each circle $\partial B_r(0)$, given $r > 0$ there exists some angle θ_r such that $u(r, \theta_r) = c_1$. Without loss of generality, we assume $\theta_r = 0$. Then, applying the Harnack inequality for $B_{r/2}(r, 0)$, we have $C_3^{-1} c_1 \le u(r,\theta) \le C_3^1 c_1$ holds in $|\theta| \le \frac{\pi}{10}$ for some C_3. We apply the same argument on $B_{r/2}(r\cos\frac{\pi}{10}, r\sin\frac{\pi}{10})$ so that we have $C_3^{-2} c_1 \le u(r,\theta) \le C_3^2 c_1$ holds in $\theta \le [-\frac{\pi}{10}, \frac{2\pi}{10}]$. We iterate this process finite times to obtain $C_4^{-1} c_1 \le u \le C_4 c_1$ on $\partial B_r(0)$ for some C_4 which is independent of r. Namely, we have $u \le C_5$ in $\mathbb{R}^2 \setminus \{0\}$.

Now, we recall $a_n(r) = (\pi)^{-1} \int_0^{2\pi} u(r,\theta)\cos(n\theta) d\theta$ which satisfies $a_n'' + r^{-1} a_n' - n^2 r^{-2} a_n = 0$. Solving ODEs yields $a_n(r) = c_{1,n} r^{-n} + c_{2,n} r^n$. However, $|a_n| \le C_5$ holds for all $r > 0$ and thus $a_n = 0$. Similarly, $b_n(r) = (\pi)^{-1} \int_0^{2\pi} u(r,\theta)\sin(n\theta) d\theta = 0$. Therefore, $u(r,\theta) = a_0(r) = c_1$. □

Second proof of Problem 7. We define $v : \mathbb{R}^2 \to \mathbb{R}$ by

$$v(y_1, y_2) = u(e^{y_1}\cos y_2, e^{y_1}\sin y_2). \tag{11}$$

Then, we can directly compute $\Delta v = e^{2y_1} \Delta u = 0$. Hence, v is an entire positive harmonic function. Therefore, by the Liouville theory v is a constant. Namely, u is a constant. □

Proof of Problem 9. Since $u(x_1, 0) = 0$, we have

$$u(r\cos\theta, r\sin\theta) = \sum_{m=1}^{\infty} a_m(r)\sin(m\theta), \tag{12}$$

where

$$a_m(r) = \frac{2}{\pi}\int_0^{\pi} u(r\cos\theta, r\sin\theta)\sin(m\theta)d\theta. \tag{13}$$

Since u is harmonic, we have

$$a_m'' + \tfrac{1}{r}a_m' - \tfrac{m^2}{r^2}a_m = 0, \tag{14}$$

and thus

$$a_m(r) = b_m r^{-m} + c_m r^m, \tag{15}$$

for some constant b_m, c_m.

However, we have $|u(x)| \leq x_2 \leq r$ and thus

$$|a_m(r)| \leq \frac{2}{\pi}\int_0^{\pi} r d\theta = 2r. \tag{16}$$

Hence, $a_1(r) = c_1 r$ and $a_m(r) = 0$ for $m \geq 2$. Namely,

$$u = \tfrac{2}{\pi}c_1 r\sin\theta = \tfrac{2c_1}{\pi}x_2. \tag{17}$$

Since $|u| \leq x_2$, there exists some $c \in [-1, 1]$ such that $u = cx_2$. □

Proof of Problem 10. As like the problem set 2, by the divergence theorem and the Hölder inequality, $E(t) = \int_\Omega u(x,t)dx$ (where $\Omega = (-1,1)^n$) satisfies

$$E' = \int_\Omega u_t dx = \int_\Omega \Delta u + u^2 dx = \int_\Omega u^2 dx \geq \left(\int_\Omega dx\right)^{-1} E^2 = E^2. \tag{18}$$

Suppose that $E(T) < 0$ holds at some $T \in \mathbb{R}$. Then, $E(t) \leq E(T) < 0$ for all $t \leq T$. Hence, we can divide (18) by E^2 to obtain $-(E^{-1})' \geq 1$ for $t \leq T$. This implies

$$-E^{-1}(T) = -E^{-1}(t) - \int_t^T (E^{-1})'(s)ds \geq \int_t^T ds = T - t. \tag{19}$$

Passing $t \to -\infty$ yields a contradiction, namely $E(t) \geq 0$ for all $t \in \mathbb{R}$.

Next, we suppose $E(T) > 0$ holds at some $T \in \mathbb{R}$. Then, $E(t) \geq E(T) > 0$ for all $t \geq T$. Hence, we can divide (18) by E^2 to obtain $-(E^{-1})' \geq 1$ for $t \geq T$. This implies

$$E^{-1}(T) = E^{-1}(t) - \int_T^t (E^{-1})'(s)ds \geq \int_T^t ds = t - T. \tag{20}$$

Passing $t \to \infty$ yields a contradiction, namely $E(t) \leq 0$ for all $t \in \mathbb{R}$.

In conclusion, $E(t) = 0$ holds for all $t \in \mathbb{R}$. Hence, (18) implies

$$0 = E' = \int_\Omega u^2 dx, \tag{21}$$

and therefore $u = 0$ in Ω. Since u is periodic, $u = 0$ in $\mathbb{R}^n \times \mathbb{R}$. □

First proof of Problem 11. Let $K = \sup \sqrt{f}$ and given $\epsilon > 0$ define

$$w_\epsilon(x) = u(x) + \tfrac{1}{2}(K+\epsilon)(1+\epsilon-|x|^2). \tag{22}$$

We claim that $w_\epsilon \geq 0$ holds in $B_1(0)$. If not, there exists $x_0 \in B_1(0)$ such that $\inf w_\epsilon = w_\epsilon(x_0)$, because $w_\epsilon = \frac{1}{2}(K+\epsilon)\epsilon > 0$ on $\partial B_1(0)$.

Since $\nabla^2 u(x_0)$ is a symmetric matrix, there exist two unit orthogonal eigenvectors v_1, v_2 and corresponding eigenvalues λ_1, λ_2. In particular, the strict convexity implies $\lambda_1, \lambda_2 > 0$. Since w_ϵ attains its minimum at the interior point x_0, we have

$$0 \leq v_i^T \nabla^2 w_\epsilon(x_0) v_i = v_i^T \left(\nabla^2 u(x_0) - (K+\epsilon)I\right) v_i = \lambda_i - (K+\epsilon), \tag{23}$$

for each $i = 1, 2$, where I is the identity matrix. This yields a contradiction as follows.

$$f(x_0) = \det(\nabla^2 u)(x_0) = \lambda_1\lambda_2 \geq (K+\epsilon)^2 > K^2 = \sup f. \tag{24}$$

Namely, $w_\epsilon \geq 0$ holds, and thus passing $\epsilon \to 0$ complete the proof. □

Second proof of Problem 11. We recall w_ϵ and its interior maximum point x_0 in the first proof. Since $\nabla^2 w_\epsilon(x_0)$ is semi-positive definite, we have

$$0 \geq \det(\nabla^2 w_\epsilon(x_0)) = \det(\nabla^2 u(x_0) - (K+\epsilon)I) \tag{25}$$

$$= \det(\nabla^2 u(x_0)) - (K+\epsilon)\Delta u(x_0) + (K+\epsilon)^2. \tag{26}$$

On the other hand, we have $\frac{1}{2}\Delta u \geq (\det \nabla^2 u)^{\frac{1}{2}} = \sqrt{f}$. Hence,

$$0 \geq f(x_0) - 2\sqrt{f(x_0)}(K+\epsilon) + (K+\epsilon)^2 = (K+\epsilon-\sqrt{f(x_0)})^2 \geq \epsilon^2. \tag{27}$$

Namely, we have $w_\epsilon \geq 0$ by the contradiction above.

□

Proof of Problem 12. Without loss of generality, we assume $\Omega \subset B_R(0) \setminus B_1(0)$. We recall the heat kernel

$$K(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}, \tag{28}$$

which is continuous in $\mathbb{R}^n \times [0,T] \setminus (0,0)$, $K(x,0) = 0$ for $x \neq 0$, and $K > 0$ for $t > 0$. By the continuity, there exists some constant M such that

$$M = \sup_{\Omega \times [0,T]} K(x,t). \tag{29}$$

On the other hand, $g > 0$ on $\partial\Omega$ implies that there exists some $\epsilon > 0$ such that $\inf_{\partial\Omega} g = \epsilon$. Hence, $v(x,t) = \epsilon M^{-1} K(x,t)$ satisfies $v \leq u$ on $\partial_p Q_T$. Therefore, by the (weak) maximum principle we have $u \geq v$ in Q_T. In particular, we have $v = \epsilon M^{-1} v > 0$ for $t > 0$, and therefore $u > 0$ for $t > 0$. ∎